DECOMPOSITION OF THE LINE INTO COUNTABLY-MANY MEASURE-THEORETIC DENSE SETS

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Like the warp and woof of a piece of cloth, two sets may be thoroughly intermingled. But how intermingled can disjoint sets be? With this in mind we ask the following question:

> Can $\mathbb{R}^n$ be decomposed into countably-many (or even just two) disjoint Lebesgue measurable sets such that the intersection of any one of these sets with *any* continuous (non-constant) curve has positive one-dimensional Hausdorff measure (or, at least, positive Hausdorff dimension)? (For the definition of Hausdorff measure and Hausdorff dimension, see, e.g., [1].)

In this note we show that the one-dimensional case is true, that is, we show that the real line $\mathbb{R}$ can be decomposed into countably-many Lebesgue measurable sets such that the intersection of any of these sets with *any* open interval has positive measure. (Decomposition of $\mathbb{R}$ into two such sets was posed as a problem in [2, p. 59].)

We call a measurable subset E of an interval I an m-dense set (with respect to I) if for any open interval $I_1 \subset I$ we have $0 < m(E \cap I_1)$, where m represents one-dimensional Lebesgue measure. The one-dimensional problem is then whether $\mathbb{R}$ can be expressed as the union of mutually-disjoint m-dense sets. It suffices to carry out the construction of countably-many m-dense sets on the unit interval $[0, 1)$, then extend these sets to be periodic of period one. Suppose the following statement is true:

(1) Any set B which is m-dense with respect to the unit interval is the union of two disjoint m-dense sets C and D with $m(D) = m(B)/2$.

Then, starting with the unit interval and iterating this decomposition, we obtain

$$[0,1) = \bigcup_{1}^{n} A_i \cup B_n$$