A CHALLENGING AREA PROBLEM REVISITED

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Some years ago a problem was proposed in the *American Mathematical Monthly* [1] for which the editors received no correct solutions before the deadline. Although eventually a solution was published (under the title "One Tough Area Problem" [2]), it is relatively involved. I would like to present a quite different, simpler solution.

The problem is to find the area of the convex planar region

$$R = \{P : PA + PB + PC \leq 2a\},$$

where ABC is an equilateral triangle of perimeter $3a$.

For convenience we take $a = 1$. We start by imposing a rectangular coordinate system in which the coordinates of A, B, C are $(-1/2, 0)$, $(1/2, 0)$, and $(0, \sqrt{3}/2)$ respectively. As mentioned in [2], the convexity of R is relatively easy to show using the triangle inequality. Let ∂R denote the boundary of R. Clearly A, B and C are on ∂R. We may deduce that the portion of ∂R in quadrant I is a convex curve connecting C and B. A parameterization of this curve may be obtained by constructing a circle of radius r, $0 \leq r \leq 1$, with center C; and an ellipse with foci A and B, and major axis of length $2 - r$. If P is the point of intersection in quadrant I, then $PC = r$ and $PA + PB = 2 - r$, so that $PA + PB + PC = 2$. See the figure below. As r goes from 0 to 1, P travels along ∂R from C to B. The coordinates (x, y) of P can be found by solving the system

$$x^2 + (y - \sqrt{3}/2)^2 = r^2 \tag{1}$$

$$\frac{x^2}{\left(\frac{2-r}{2}\right)^2} + \frac{y^2}{\left(\frac{2-r}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = 1. \tag{2}$$