EVALUATION OF A FAMILY OF SUMMATIONS

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The purpose of this paper is to evaluate

$$S = S(p) = \sum_{n=1}^{2p-1} \left(2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n} + \frac{n\pi}{2p} \right) \right) \tag{1}$$

where p is any positive integer. When $n = 1$ the first term in the sum is taken to be 1 since empty products are assumed to be 1.

Lemma. If $\sin\theta \neq 0$, then

$$2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n} + \theta \right) = \frac{\sin n\theta}{\sin\theta}.$$

Proof. Since the nth roots of unity are

$$e^{\frac{2k\pi i}{n}}$$

for $k = 0, 1, \ldots, n-1$,

$$x^n - 1 = (x-1) \prod_{k=1}^{n-1} \left(x - e^{\frac{2k\pi i}{n}} \right). \tag{2}$$