SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

37. [1991, 149] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Lines l_1 and l_2 are concurrent at O. Let $\{a_i\}$ be a sequence of points on l_1 and $\{b_i\}$ be a sequence of points on l_2 such that

$$d(O, a_1) = d(a_i, a_{i+1}) = d(O, b_1) = d(b_i, b_{i+1}) > 0$$

for $i = 1, 2, 3, \ldots$. If M_i is the midpoint of the line segment $\overline{a_i b_i}$, prove that the points M_i are collinear.

Solution I by Andrea Rothbart, Webster University, St. Louis, Missouri. This solution uses a vector approach. For simplicity, let $\overrightarrow{x}$ stand for the vector $\overrightarrow{Oa_1}$ and $\overrightarrow{y}$ stand for the vector $\overrightarrow{Ob_1}$. Then, $\overrightarrow{Ob_n} = n\overrightarrow{y}$ and $\overrightarrow{Oa_n} = n\overrightarrow{x}$. Also,

$$\overrightarrow{b_n a_n} = \overrightarrow{b_n O} + \overrightarrow{Oa_n} = -n\overrightarrow{y} + n\overrightarrow{x} = n(\overrightarrow{x} - \overrightarrow{y}) \ .$$

Thus,

$$\overrightarrow{OM_n} = \overrightarrow{Ob_n} + \overrightarrow{b_n M_n} = n\overrightarrow{y} + \frac{1}{2}n(\overrightarrow{x} - \overrightarrow{y}) = \frac{n}{2}(\overrightarrow{x} + \overrightarrow{y}) = n\overrightarrow{OM_1} \ .$$

Since each of the vectors $\overrightarrow{OM_n}$ is a multiple of $\overrightarrow{OM_1}$, the points M_1, M_2, $\ldots$ are collinear.

Solution II by Seung-Jin Bang, Seoul, Republic of Korea. We may assume that the line l_1 is the x-axis, the lines l_1 and l_2 are concurrent at $O = (0, 0)$, and $d(O, a_1) = a > 0$. Then we have $a_i = (ia, 0)$.

Case I. The lines l_1 and l_2 are perpendicular. Then $b_i = (0, ia)$. It follows that

$$M_i = \frac{1}{2}(a_i + b_i) = \frac{ia}{2}(1, 1)$$

and the points M_i lie on the line $y = x$.

Case II. The equation of the line l_2 is $y = kx$, $k = \tan\theta$ $(0 < \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta < \pi)$. Then

$$b_i = (ia\cos\theta, ia\sin\theta) \ .$$