INTEGRATING POWERS OF TRIGONOMETRIC FUNCTIONS

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The objective of this article is to show that the indefinite integrals of the six trigonometric functions raised to an even or odd power can be expressed in a closed form. Handbooks of mathematics and calculus textbooks show the integrals of the trig functions to the nth power in the form of reduction formulas. Some computer programs on the market will evaluate these integrals but not in closed form. If the reader does not have access to one of these commercial programs then you can use the following ideas to write your own programs. First, in order to obtain our main objective, this article shows how to find closed forms for the integrals of

$$\frac{dx}{(a^2 \pm x^2)^n} \ .$$

Then with certain substitutions, closed forms of trigonometric integrals can be found.

Theorem 1. The following formula holds true:

$$\int \frac{dx}{(a^2+x^2)^n} = \binom{2n-2}{n-1} \frac{\tan^{-1}\frac{x}{a}}{2^{2n-2}a^{2n-1}}$$

$$(1) \qquad - \sum_{k=0}^{n-2} \frac{\binom{n+k-1}{k}\left[\sum_{j=1}^{n-k-1}\binom{n-k-1}{j} a^{n-k-1-j} x^j i^{j+1}\right]}{(n-k-1)a^{n+k}2^{n+k-1}(x^2+a^2)^{n-k-1}} + C \ , \quad a \neq 0 \ , \quad i = \sqrt{-1}$$

where $j = 1, 3, 5, \ldots$ only.

Proof. Consider the following two integrals,

$$(2) \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \ , \quad a \neq 0$$