SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

25. [1990, 140] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let P be the free product with amalgamations of any collection $\{C_\gamma\}_{\gamma\in\Gamma}$ of infinite cyclic groups, where Γ is an indexing set of cardinality greater than one.

A. Construct an element $h \in P$ and a subgroup $T < P$ such that $|P : T|$ is infinite but $|P :< h, T > |$ is finite.

B. Is h uniquely determined? Is T uniquely determined?

Solution by the proposer.

Let $C_\gamma =< c_\gamma >$, let $H_\gamma =< c_\gamma^{m_\gamma} >$, where m_γ are natural numbers, greater than one, and let

$$P = \star_{\gamma\in\Gamma}(C_\gamma; H_\gamma)$$

with amalgamated subgroup H.

A. Consider an arbitrary element

$$h = c_\gamma^{qm_\gamma} \quad (\gamma \in \Gamma)$$

of H, where q is a natural number. Now, let

$$T =< c_\gamma^{p_\gamma} : \gamma \in \Gamma > ,$$

where p_γ are prime numbers greater than one and $(p_\gamma, qm_\gamma) = 1$, for all $\gamma \in \Gamma$. We claim that this h is the required element and also the subgroup T defined as above is the required subgroup. (Note that if Γ is finite and p is any prime number greater than one, such that $(p, qm_\gamma) = 1$, then

$$T =< c_\gamma^{p} : \gamma \in \Gamma >) .$$

Proof of claim: $(p_\gamma, qm_\gamma) = 1$ implies that there are integers r_γ and s_γ such that

$$r_\gamma p_\gamma + s_\gamma qm_\gamma = 1 .$$

Thus,

$$c_\gamma = c_\gamma^{r_\gamma p_\gamma + s_\gamma qm_\gamma}$$