# SEQUENCES, MATHEMATICAL INDUCTION, AND THE HEINE-BOREL THEOREM

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The usual proof of the Heine-Borel Theorem as in [1] and [2], always makes use of the Bolazano-Weierstrass Theorem and/or Cantor's Intersection Theorem. But if the understanding of these theorems exhausts the students' mathematical knowledge, then an understanding of the proof is lost.

The following method of proof, which is simpler but written in the same spirit as Ross's work in [3], uses only results from the theory of sequences and mathematical induction. This not only allows us to avoid the use of these "big" theorems, but also strengthens the students' understanding of sequences and mathematical induction.

Definition. An $n-$dimensional open interval $I$ in $\Re^n$ is defined by the set

$$I = \{(x_1, x_2, \cdots, x_n) | a_i < x_i < b_i; a_i, x_i, b_i \in \Re; 1 \leq i \leq n\} \ .$$

Similarly, an $n-$dimensional closed interval $J$ in $\Re^n$ is defined by the set

$$J = \{(x_1, x_2, \cdots, x_n) | \ a_i \leq x_i \leq b_i; a_i, x_i, b_i \in \Re; 1 \leq i \leq n\} \ .$$

Definition. A set $O$ in $\Re^n$ is said to be open if for each $x$ in $O$, there exists an $n$-dimensional open interval $I$ such that $x \in I \subseteq O$. A set $C$ in $\Re^n$ is closed if $\Re^n - C$ is open.

Remark 1. Every $n$-dimensional closed interval in $\Re^n$ is closed.

Definition. A sequence $s$ in a nonempty set $S$ is a function whose domain is the set of natural numbers and whose range is a subset of $S$.

Throughout this paper $N$ is used to denote the set of natural numbers and $R(s)$ is used to denoted the range of a sequence $s$. A sequence $s$ is said to be infinite if $R(s)$ is an infinite set.

Definition. Let $s$ be a sequence in $\Re^n$. A sequence $t$ is called a subsequence of $s$ if there is an increasing function $\phi$ from $N$ to $N$ such that $t = s \circ \phi$.