ON UNIT GROUPS OF EXTENSION RINGS: AN EXAMPLE

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The following is a well known result of algebraic number theory [1] or [2].

Theorem (Dirichlet). Suppose $Q[\alpha]$ is a finite degree extension field over the field Q of rational numbers and R is the integral closure in $Q[\alpha]$ of the ring Z of integers. Let r and $2s$ be the numbers of real and nonreal embeddings, respectively, of $Q[\alpha]$ into the field of complex numbers. Then $U(R)$, the group of units of R, can be written as a direct product $U(R) \cong G \times H$ where G is a finite group and H is a free abelian group of rank $r+s-1$.

One peculiarity of the above result is that while it appears to be entirely algebraic, it seems that all known proofs involve some analysis. In fact, there does not seem to exist an entirely algebraic proof of the weaker conclusion that $U(R)$ has finite rank. Following this same line of questioning, a friend of the author recently posed the following problem.

Conjecture. Let R be an integral domain with field of fractions K and suppose that $U(R)$, the unit group of R, has finite rank. Suppose that F is a finite degree extension field over K and that S is the integral closure in F of R. Then $U(S)$, the unit group of S, has finite rank.