SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

10. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Show

$$\sum_{\substack{k=1\\(k,45)=1}}^{45} \sin^8 \frac{k\pi}{45} = \frac{51}{8} \ .$$

Solution by the proposers.

Let $w = \exp \frac{\pi i}{n}$. Then

$$\begin{aligned}
\sum_{k=1}^{n} \sin^{4m} \frac{k\pi}{n} &= \sum_{k=1}^{n} \left(\frac{w^k - w^{-k}}{2i} \right)^{4m} \\
&= \frac{1}{2^{4m}} \sum_{k=1}^{n} \sum_{j=0}^{4m} (-1)^j \binom{4m}{j} \left(w^k\right)^{4m-j} \left(w^{-k}\right)^j \\
&= \frac{1}{2^{4m}} \sum_{k=1}^{n} \sum_{j=0}^{4m} (-1)^j \binom{4m}{j} \left(w^{4m-2j}\right)^k \\
&= \frac{1}{2^{4m}} \sum_{j=0}^{4m} (-1)^j \binom{4m}{j} \sum_{k=1}^{n} \left(w^{4m-2j}\right)^k \ .
\end{aligned}$$

Now

$$\sum_{k=1}^{n} \left(w^{4m-2j}\right)^k = \begin{cases} n, & \text{if } w^{4m-2j} = 1, \\ \frac{w^{4m-2j}(1-(w^n)^{4m-2j})}{1-w^{4m-2j}} = 0, & \text{if } w^{4m-2j} \neq 1. \end{cases}$$

But $w^{4m-2j} = 1$ iff $\frac{2m-j}{n}$ is an integer. Also, if $n \mid 2m - j$, then there exists an integer k such that $2m - j = kn$. Therefore,

$$0 \le j = 2m - kn \le 4m$$