SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

6. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Prove

$$\sum_{n\le x}\frac{1}{3n-2}=\frac{1}{3}\log(3x-2)+\frac{1}{6}\log 3+\frac{\pi}{6\sqrt{3}}+\frac{\gamma}{3}+O\left(\frac{1}{x}\right),$$

where log is the natural log and γ is Euler's constant.

Solution by the proposers.

We start with the following lemma.
Lemma.

$$\int_{\frac{1}{3}}^{\infty}\frac{u-[u+\frac{2}{3}]}{u^2}\,du=1-\gamma-\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\log 3\ .$$

Proof.

$$\int_{\frac{1}{3}}^{\infty}\frac{u-[u+\frac{2}{3}]}{u^2}\,du=\int_{\frac{1}{3}}^{1}\frac{u-[u+\frac{2}{3}]}{u^2}\,du+\sum_{n=1}^{\infty}\int_{n}^{n+1}\frac{u-[u+\frac{2}{3}]}{u^2}\,du$$

$$=\int_{\frac{1}{3}}^{1}\frac{u-1}{u^2}\,du+\sum_{n=1}^{\infty}\left(\int_{n}^{n+\frac{1}{3}}\frac{u-[u+\frac{2}{3}]}{u^2}\,du+\int_{n+\frac{1}{3}}^{n+1}\frac{u-[u+\frac{2}{3}]}{u^2}\,du\right)$$

$$=\log u+\frac{1}{u}\Big|_{\frac{1}{3}}^{1}+\sum_{n=1}^{\infty}\left(\int_{n}^{n+\frac{1}{3}}\frac{u-[u]}{u^2}\,du+\int_{n+\frac{1}{3}}^{n+1}\frac{u-[u]-1}{u^2}\,du\right)$$

$$=\log 3-2+\sum_{n=1}^{\infty}\left(\int_{n}^{n+1}\frac{u-[u]}{u^2}\,du-\int_{n+\frac{1}{3}}^{n+1}\frac{du}{u^2}\right)$$

$$=\log 3-2+\int_{1}^{\infty}\frac{u-[u]}{u^2}\,du+\sum_{n=1}^{\infty}\frac{1}{u}\Big|_{n+\frac{1}{3}}^{n+1}$$