THE VOLUME OF AN n-SIMPLEX WITH MANY EQUAL EDGES

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It is well known that the volume of a regular n-simplex with edge length s is

$$\frac{s^n}{n!}\sqrt{\frac{n+1}{2^n}}\ .$$

But suppose one edge has length b and all the other edges have length a. Is there a simple formula for the volume of the simplex in that case? What if all the edges incident at a given vertex have length b and all the other edges have length a?

It is these questions that motivated the investigation that led to the following result:

Theorem. Let K be an n-simplex in E^n. Suppose the vertices of K are colored with r colors, c_1, c_2, …, c_r $(1 \le r \le n+1)$. Let the number of vertices colored c_i be m_i $(1 \le m_i \le n+1)$. It is given that if an edge has both its vertices the same color, c_i, the length of that edge is a_i. If the two vertices of an edge have different color, the edge has length s. Then the volume of K is

$$\frac{1}{n!\,2^{\frac{n}{2}}}\prod_{i=1}^{r} a_i^{m_i-1}\sqrt{(-1)^{r+1}\left(\prod_{i=1}^{r}\big((m_i-1)a_i^2-m_is^2\big)\right)\sum_{i=1}^{r}\frac{m_i}{(m_i-1)a_i^2-m_is^2}}\ .$$