No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

157. [2005, 194] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let a, x be real numbers such that $1 < a < x$. Prove that

$$\left(\sum_{k=1}^{n} \log^{-1}_{a^{k(k+1)}} x\right)\left(\log_a \sqrt[n+1]{x^n}\right) \geq n^2.$$

Solution by Joe Flowers, St. Mary's University, San Antonio, Texas. We have

$$\log_a \sqrt[n+1]{x^n} = \frac{n}{n+1}\log_a x$$

and

$$\log_{a^{k(k+1)}} x = \frac{\log_a x}{\log_a a^{k(k+1)}} = \frac{\log_a x}{k(k+1)},$$

so

$$\log^{-1}_{a^{k(k+1)}} x = \frac{k(k+1)}{\log_a x}.$$

Therefore,

$$\left(\sum_{k=1}^{n} \log^{-1}_{a^{k(k+1)}} x\right)\left(\log_a \sqrt[n+1]{x^n}\right) = \left(\frac{1}{\log_a x}\sum_{k=1}^{n} k(k+1)\right)\left(\frac{n}{n+1}\log_a x\right)$$

$$= \frac{n}{n+1}\left(\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right) = \frac{n}{n+1}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= n^2 \cdot \frac{n+2}{3} \geq n^2,$$