ON THE DENSITIES OF SOME SUBSETS OF INTEGERS

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In this note, we prove two conjectures concerning densities of subsets of positive integers suggested in [1] and [2], respectively. Throughout this paper, we use p and q for prime numbers and x for a large positive real number. If $\mathcal{A} \subset \mathbb{N}$ is a subset of the positive integers, we write $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. We use the Vinogradov symbols $\ll$ and $\gg$, and the Landau symbols O and o with their usual meanings. Namely, we say that $f(x) \ll g(x)$, or that $f(x) = O(g(x))$, if the inequality $|f(x)| < cg(x)$ holds with some positive constant c for all sufficiently large x. The notation $g(x) \gg f(x)$ is equivalent to $f(x) \ll g(x)$, while $f(x) = o(g(x))$ means that $f(x)/g(x)$ tends to zero when x tends to infinity. We use $\log x$ for the natural logarithm of x.

1. Sigma-Primes. Following [1], a positive integer n is called a *sigma-prime* if n and $\sigma(n)$ are coprimes, where $\sigma(n)$ is the sum of the divisors of n. Let $\mathcal{SP}$ be the set of all sigma-primes. It was conjectured in [1] that $\mathcal{SP}$ is of asymptotic density zero. Here, we prove this conjecture.

Theorem 1. The inequality

$$\#\mathcal{SP}(x) \ll \frac{x}{\log\log\log x}$$

holds for all $x > e^e$.

Proof. Let x be a large positive real number. Lemma 4 in [5] asserts that there exists an absolute constant c_1 such that $\sigma(n)$ is divisible by all primes

$$p < y := c_1 \frac{\log\log x}{\log\log\log x}$$

for all $n < x$ except for a subset of such n of cardinality $O(x/\log\log\log x)$. Thus,

$$\#\mathcal{SP}(x) \le \#\{n \le x \ : \ \gcd(n, p) = 1 \ \text{ for all } \ p \le y\} + O\left(\frac{x}{\log\log\log x}\right). \qquad (1)$$