SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

173. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Show that

$$\sum_{n=1}^{\infty} \frac{x_n}{x_{n-1}} = \frac{7}{2},$$

provided

$$x_{n-1}(x_{n-2}^2 + x_{n-1}x_{n-3}) - 6x_{n-3}(x_{n-1}^2 - x_n x_{n-2}) = 0, \quad n \geq 3,$$

and $x_0 = x_1 = x_2 = 1$.

Solution by Panagiotis T. Krasopoulos, Athens, Greece. First, let us observe that from the statement of the problem it is assumed implicitly that $x_k \neq 0$ for any $k \geq 0$. This fact will be proved in the process of the following proof.

Let us assume that $x_k \neq 0$ for any $0 \leq k \leq n-1$. We divide the given equation by the product $x_{n-1}x_{n-2}x_{n-3}$ and we define $a_n = x_n/x_{n-1}$, so we obtain

$$a_{n-2} + a_{n-1} - 6a_{n-1} + 6a_n = 0 \text{ if and only if } 6a_n - 5a_{n-1} + a_{n-2} = 0,$$

where $n \geq 3$ and $a_1 = a_2 = 1$. This is a linear homogeneous difference equation with constant coefficients and can be solved directly by using its characteristic equation. After some algebraic calculations we have

$$a_n = 8\left(\frac{1}{2}\right)^n - 9\left(\frac{1}{3}\right)^n \text{ for } n \geq 1.$$

It can easily be seen that $\frac{8}{9} > \left(\frac{2}{3}\right)^n$ for $n \geq 1$ and so $a_n > 0$. Since $a_n > 0$ and $x_0 = x_1 = x_2 = 1 > 0$, by induction we obtain that $x_k > 0$ for any