DIOPHANTINE EQUATIONS, FIBONACCI HYPERBOLAS, AND QUADRATIC FORMS

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1.   Introduction. In *Mathematical Diversions* [4], Hunter and Madachy ask for the ages of a boy and his mother, given the following:

His mother's age and his, when multiplied together, come to one more than the square of their difference.

We used this problem in a contest for our students. While reviewing the solutions submitted, we discovered some oddities about this question and its corresponding Diophantine equation. All integer solutions to the equation, and those of an incorrect formulation of the equation, consist of Fibonacci numbers. Furthermore, there is an interesting duality between the solutions to the correct and incorrect formulations.

2. The Equations. Let x and y represent the ages of the boy and his mother, respectively. The question leads to the equation

$$xy = 1 + (x - y)^2,$$

which simplifies to

$$x^2 - 3xy + y^2 = -1. \tag{1}$$

One of the solutions submitted had the 1 on the wrong side of the equation,

$$xy + 1 = (x - y)^2,$$

which simplifies to

$$x^2 - 3xy + y^2 = 1. \tag{2}$$