RETURNS TO THE ORIGIN FOR RANDOM WALKS ON $\mathbb{Z}$ REVISITED

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Chrysafi and Bradley [1] considered symmetric random walks, defined as follows: let $X_k, k = 1, 2, \dots$ be independent and identically distributed random variables with $\mathbb{P}\{X_k = 1\} = \mathbb{P}\{X_k = -1\} = \frac{1}{2}$. Then

$$S_m = \sum_{k=1}^{m} X_k \quad \text{with} \quad S_0 = 0$$

is a simple random walk starting at 0. The authors considered only walks of even length $m = 2n$ and were interested in the random variable $R = R_n$, defined to be the *number of returns to the origin* in a walk of length $2n$, i.e., the number of times $S_i = 0$ happens, for $i = 1, \dots, 2n$. They computed moments up to $\mathbb{E}[R^6]$ and asked for a closed formula for $\mathbb{E}[R^k]$ and also whether $\mathbb{E}[R^k] \sim c_k n^{k/2}$ holds.

The answers to these questions can be found in [4] as opposed to [1]. There, the *factorial moments* $\mathbb{E}[R^k]$ were computed. We state the formula only for even n:

$$\mathbb{E}[R_n^k] = k! \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{\frac{i}{2} + n}{n}.$$

Ordinary moments can be recovered from these formulae as linear combinations with *Stirling numbers of the second kind (Stirling subset numbers),* see [3].

$$\mathbb{E}[R_n^k] = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} \mathbb{E}[R_n^i].$$

To answer the asymptotics question, we use generating functions

$$\mathbb{E}[R_n^k] = 4^{-n} k! [z^n] \frac{(1 - \sqrt{1 - 4z})^k}{(1 - 4z)^{1 + \frac{k}{2}}},$$