

ADDING SOMETHING EXTRA

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*It requires more innate intellectual capacity
to dispose of this apparently childish thing
than to grasp the theory of relativity.*

E. T. Bell

1. Introduction. The “thing” referred to in the above quote is the Diophantine equation

$$y^3 = x^2 + 2$$

and to “dispose of” this thing is to find all its positive integer solutions [1]. It is clear that $x = 5$ and $y = 3$ is a solution, but Fermat, as a challenge to other mathematicians, asked for a proof that this is the *only* such solution [5]. The challenge was met, more than a hundred years later, by Euler. Euler’s short proof is easily followed by students taking a course in algebra or number theory. So what would lead E. T. Bell to make such a bold claim for the proof?

Euler’s proof is remarkable because it uses *complex numbers* to prove something about *positive integers*. In other words, extra structure is added to help unravel the secret of the equation. This principle of “adding something extra” in a proof is discussed by George Polya in his classic book *How to Solve It* [4]. So how does adding extra structure help us solve a problem? In this note, we illustrate how this principle works, then give a sketch of Euler’s proof in light of this principle. In our classes, we have found that explicitly identifying when extra structure is added in a proof gives students an appreciation for the ingenuity required to invent the proof, as well as an understanding of a principle they can recognize when they encounter its use in other proofs.

First, to see how adding something extra can help solve a problem, we consider a simple problem where this principle is clearly and strikingly used.

2. The Maddening Mouse Maze. A maze is in the shape of a square with six rooms to a side. Each room is connected by a door to each of its contiguous neighbors. The entrance and exit are at diagonally opposite corners, as shown in the figure below. A mouse entering the maze must pass through each room exactly once in order for the exit door to open. What route should the mouse take?