SOME REMARKS ON THE SUM OF AN OLD SERIES

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In this note we use some combinatorial identities to derive a formula for the sum of the series

$$S(p) = \sum_{n=0}^{\infty} (m+n+1)^p \binom{n+m}{m} x^n, \quad |x| < 1,$$

in the form $P(x)/(1-x)^{m+p+1}$, where $P(x)$ is a polynomial of degree $p-1$ with known coefficients a_j, $0 \le j \le p-1$. When specialized for $m = 0$, the resulting sum gives a formula for

$$\sum_{n=1}^{\infty} n^p x^n \quad (|x| < 1).$$

The general formula also provides an alternative method for determining the moments of a negative binomial distribution. Conversely, the negative binomial distribution can be used to find a recursive formula for the sum of the above series $S(p)$.

1. A Combinatorial Identity. In what follows we write

$$x_{(r)} = x(x-1)(x-2)\cdots(x-r+1)$$

for any real x and positive integer r, and in particular $x_{(r)} = x!/(x-r)!$ if x is also a positive integer, and $x_{(0)} = 1$, etc. Also, for a function $\psi(t)$, $\psi^{(k)}(a)$ denotes the kth derivative evaluated at a. The identities in Lemma 1 can be found in disguised forms in Feller [2]. These identities in turn imply the main identity in Lemma 2. A generating function method is used for their proofs for the sake of completeness.