SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

145. [2004, 58] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let F_n denote the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$) and let L_n denote the nth Lucas number ($L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$). Prove that

$$F_{n+1} > \frac{1}{3}\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right)^{\frac{1}{L_n - F_n}}$$

holds for all positive integer $n \geq 2$.

Solution by the proposer. It is well known [1] that for a positive integrable function defined on the interval $[a, b]$, the integral analogue of the AM-GM inequality is given by

$$A(f) = \frac{1}{b-a}\int_a^b f(x)\,dx \geq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right) = G(f). \tag{1}$$

Setting $f(x) = x$, $a = F_n$, and $b = L_n$ into (1), yields

$$\frac{1}{L_n - F_n}\int_{F_n}^{L_n} x\,dx \geq \exp\left(\frac{1}{L_n - F_n}\int_{F_n}^{L_n} \ln x\,dx\right).$$

Note that for all $n \geq 2$, $L_n - F_n > 0$. Evaluating the preceding integrals and after simplification, we obtain

$$\begin{aligned}\frac{F_n + L_n}{2} &\geq \exp\left(\frac{1}{L_n - F_n}\ln\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right) - 1\right) \\ &= \exp\left(\ln\left[\frac{1}{e}\left(\frac{L_n^{L_n}}{F_n^{F_n}}\right)\right]^{\frac{1}{L_n - F_n}}\right). \qquad (2)\end{aligned}$$