SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

143. [2003, 201; 2004, 129] *Proposed by José Luis Diaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let α, β, and γ be the angles of acute triangle ABC. Prove that

$$\frac{\cot\alpha\cot\beta}{\sqrt{1-\cot\alpha\cot\beta}}+\frac{\cot\beta\cot\gamma}{\sqrt{1-\cot\beta\cot\gamma}}+\frac{\cot\gamma\cot\alpha}{\sqrt{1-\cot\gamma\cot\alpha}}\geq\sqrt{\frac{3}{2}}.$$

Solution by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. First we notice that the expressions under the square roots are positive since

$$1-\cot\alpha\cot\beta=\frac{-\cos(\alpha+\beta)}{\sin\alpha\sin\beta}\geq 0,$$

since $\sin\alpha,\sin\beta>0$ and $\cos(\alpha+\beta)\leq 0$. (i.e. $\alpha+\beta\geq 90^\circ$; $180^\circ-\gamma\geq 90^\circ$ implies $\gamma\leq 90^\circ$). I'll make use of the following equality which holds in any triangle:

$$\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1,$$

where α, β, and γ are the angles of a given triangle ABC. Denote

$$a=\cot\alpha\cot\beta,\quad b=\cot\beta\cot\gamma,\quad c=\cot\gamma\cot\alpha.$$

Let

$$f\colon(0,1)\to\mathbb{R},\quad\text{where}\quad f(x)=\frac{x}{\sqrt{1-x}}.$$

Then

$$f'(x)=\frac{2-x}{2(1-x)^{\frac{3}{2}}}\quad\text{and}\quad f''(x)=\frac{4-x}{4(1-x)^{\frac{5}{2}}}>0.$$