ANOTHER ELEMENTARY PROOF OF THE CONVERGENCE-DIVERGENCE OF p-SERIES

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Recently Khan [2] gave a simple proof of the convergence-divergence of the p-series $\sum_{n=1}^{\infty} 1/n^p$. The divergence of this series for $p \leq 1$ was shown by contradiction while the convergence of the series for $p > 1$ was established by the boundedness of the monotonic partial sums [2]. Here, we give a more direct and very elementary proof of the same by using only the sum of a geometric series. Moreover, a telescoping method is used to find sums of some interesting series.

We use the following simple fact:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1. \tag{1}$$

We consider integers j from 2^m to $2^{m+1} - 1$ ($m = 0, 1, 2, \ldots$), and note that the number of terms are $2^{m+1} - 1 - (2^m - 1) = 2^{m+1} - 2^m = 2^m$. Then, for any p we write the p-series as

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{m=0}^{\infty} \sum_{j=2^m}^{2^{m+1}-1} \frac{1}{j^p}. \tag{2}$$

If $p > 1$, it then follows from (1) and (2) that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{m=0}^{\infty} \frac{2^m}{2^{mp}} = \sum_{m=0}^{\infty} \frac{1}{2^{m(p-1)}} = \frac{2^{p-1}}{2^{p-1} - 1},$$

and the series converges. To show the divergence for $p \leq 1$, we consider $p = 1$ first. It is clear from (2) that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{m=0}^{\infty} \sum_{j=2^m}^{2^{m+1}-1} \frac{1}{j} \geq \sum_{m=0}^{\infty} \frac{2^m}{2^{m+1}} = \infty.$$