BETWEEN CONSECUTIVE SQUARES

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The mystery of the distribution of the primes continues in its challenge to mathematicians as the twenty-first century unfolds. Among its many unfinished chapters is the tantalizing question surrounding the seeming occurrence of primes between any two squares.

The pioneering work of P. L. Tchebychef (1821–1894) in the mid-nineteenth century, building on the notes of Adrien Marie Legendre (1752–1833) and Carl Friedrich Gauss (1777–1855), was to set the stage for an ultimate and rigorous disposition of the Prime Number Theorem. Not only would he resolve Bertrand's Conjecture in the affirmative and thus provide another look at the infinitude of the primes, so too would he provide a deeper look at $\pi(x)$. Such a symbol denotes the number of primes less than or equal to x. This early analysis of $\pi(x)$ bordered closely on establishing the key limit result itself, namely,

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1.$$

Such a theorem would find its resolution in the simultaneous discoveries of Jacques Hadamard (1865–1963) and Charles de la Vallée Poussin (1866–1962). The year of discovery was 1896. Though the theorem speaks of the number of primes only in an approximate manner, it provides a powerful basis for various conjectures as diverse number classes are considered.

Speculation. Paralleling the definition of $\pi(x)$, let $\alpha_n(x)$ represent the number of nth powers less than or equal to x. As $\pi(x) > \alpha_2(x)$ for all x sufficiently large, it is tempting to conclude that between any two squares, there exists a prime. Note that $\pi(x) \approx \frac{x}{\ln x}$ for large values of x and that $\alpha_2(x) = [\sqrt{x}] \approx \sqrt{x}$. As $\ln x < \sqrt{x}$, then $\frac{x}{\ln x} > \sqrt{x}$. Capitalizing on the fact that

$$\pi(x) > \frac{x}{\ln x} > \sqrt{x} \geq [\sqrt{x}],$$

it follows that the number of primes exceeds that of the squares as the reference number x becomes large.