UTILIZING THE EXPANSION OF $P^n - Q^n$ TO INTRODUCE AND DEVELOP THE EXPONENTIAL FUNCTION

Bhamini M. P. Nayar

Recently, Bayne et al. [1, 2], have applied the identity

$$P^n - Q^n = (P - Q)\sum_{k=0}^{n-1} P^k Q^{n-1-k} \tag{1}$$

for real P, Q and positive integers n to present simple proofs of the existence of nth roots and inequalities used in real analysis. In this article the identity (1) is used to prove that f defined by

$$f(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

is a real-valued continuous function onto the positive reals with the collection of reals as its domain, and to establish some properties of f, including $f(x+y) = f(x)f(y)$, $f(0) = 1$ and an elegant proof that $f' = f$ where f' represents the derivative function for f. The equation $f(r) = (f(1))^r$ is shown to hold for rational r. This motivates the notation $f(x) = (f(1))^x = e^x$ and calling f the exponential function.

As in [4], the exponential function is often introduced as the inverse of the logarithmic function which is defined as

$$\int_1^x \frac{1}{t}dt.$$

Later, when convergence of sequences is studied, e^x is proved to be the limit of the sequence $(1 + \frac{x}{n})^n$. There again the logarithmic function is used. Dieudonné [3] introduced the logarithmic function by proving that

For any $a > 1$, there is a unique increasing continuous function g of the positive reals into the reals such that $g(xy) = g(x) + g(y)$ and $g(a) = 1$.