FUNCTIONS AS SUMS OF ODD AND EVEN FUNCTIONS

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1. Introduction. It is well-known that every function mapping the real numbers into the real numbers can be written as the sum of an odd function and an even function. One only needs to use the additive properties of the real numbers to show this. What other groups have the property that every function on the group can be written as the sum of an odd function and an even function? To answer this, we first need to establish our basic definitions.

Let $(G, +)$ be a group, written additively, but not necessarily abelian. A function $f\colon G \to G$ is *odd* if $f(-x) = -f(x)$ for all $x \in G$ and *even* if $f(-x) = f(x)$ for all $x \in G$. We say the function $f\colon G \to G$ *splits* if f can be written as the sum of an odd function and an even function on G. Recall that a group has finite exponent if the least common multiple of the orders of all the elements of G is finite. Certainly every finite group has finite exponent, but the converse is not true. For example, the group of polynomials $(\mathbb{Z}_m[x], +)$ is infinite with finite exponent. In this paper we determine all groups G with finite exponent for which every function on G splits.

2. Main Results. Certainly if $G = \{0\}$, then every function on G splits since the zero function is both even and odd. We also can quickly identify a class of groups over which every function will split.

Lemma 1. Let G be a 2-group of exponent 2. Then every function on G can be written as the sum of an odd function and an even function.

Proof. If G has exponent two, then $a + a = 0$ and $a = -a$ for all $a \in G$. So if $f\colon G \to G$ is a function and $a \in G$, then $f(a) = f(-a)$ and f is an even function. Since the zero function 0 is odd, then $f = 0 + f$ and f splits.

In the case that G is a group of exponent two, the representation of a function $f\colon G \to G$ as the sum of an odd function and an even function is not unique. In the proof above, we showed that $f = 0 + f$ where 0 is an odd function and f is an even function. But since $x = -x$ for all $x \in G$, then $f(-a) = f(a) = -f(a)$ and f is an odd function. Since the zero function 0 is even, then $f = f + 0$ and f splits in two different ways.

Henceforth, we assume that G is a group with at least one element of order greater than two.

We say $g \in G$ is (*uniquely*) *halvable* if $g = x + x$ for some (unique) $x \in G$, and the group G is (*uniquely*) *halvable* if every element in G is (uniquely) halvable.

We now state our main result.