INFINITELY MANY COMPOSITE NSW NUMBERS: AN INDUCTIVE PROOF

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1. Motivation. The NSW numbers were introduced approximately 20 years ago [3] in connection with the order of certain simple groups. These are the numbers f_n which satisfy the recurrence

$$f_{n+1} = 6f_n - f_{n-1} \tag{1}$$

with initial conditions $f_1 = 1$ and $f_2 = 7$.

In recent years, these numbers have been studied from a variety of perspectives [1, 2]. Moreover, the author, in collaboration with Hugh Williams, has proven that there are infinitely many composite NSW numbers [4] as requested in [1]. The goal of this note is to provide a purely inductive proof of the main theorem in [4]. We restate it here.

Theorem 1.1. For all $m \geq 1$ and all $n \geq 0$, $f_m | f_{(2m-1)n+m}$.

2. The Necessary Tools. To prove Theorem 1.1, we need to develop a few key tools.

Proposition 2.1. For all integers $a, b \geq 0$, and for all $1 \leq j \leq a+b-2$, we have

$$f_{a+b} = s_{j+1} f_{a+b-j} - s_j f_{a+b-j-1} \tag{2}$$

where

$$s_j = \sum_{i=1}^{j} (-1)^{i+j} f_i.$$

Proof. We prove this proposition using induction on j. First, when $j = 1$, the right hand side of (2) is $(f_2 - f_1) f_{a+b-1} - f_1 f_{a+b-2}$ or $6f_{a+b-1} - f_{a+b-2}$, which equals f_{a+b} thanks to (1).

Next, we assume

$$f_{a+b} = s_{j+1} f_{a+b-j} - s_j f_{a+b-j-1}$$