ANOTHER LOWER BOUND FOR $\frac{\sin \mathbf{x}}{\mathbf{x}}$

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The limit

$$\lim_{x\to 0}\frac{\sin x}{x} \tag{$*$}$$

allows us to compute the derivatives of $\sin x$ and $\cos x$. As pointed out by Professor Krantz [1], however, the way it is calculated in most calculus books in print follows a circular argument. In fact, it is customary to use the Pinching Theorem by giving an upper and lower bound for $\frac{\sin x}{x}$. The upper bound

$$\frac{\sin x}{x} \leq 1$$

is trivial as it can be found by comparing the length of a chord with the length of the corresponding arc of a circle. The lower estimate

$$\frac{\sin x}{x} \geq \cos x$$

is usually done assuming the knowledge of the area of a sector, which in turn depends on knowing the area of a circle. To know the area of a circle, however, most calculus books make use of the limit ($*$). To avoid this circular argument, Professor Krantz proposed in [1] the alternative lower bound

$$\frac{\sin x}{x} > \frac{1}{1+\tan x}.$$

This estimate avoids the use of the area of a circle and uses only the definition of arc length and some elementary geometry. A different estimate was given in [2]. The authors proved the inequalities

$$\sin x \leq x \leq \tan x$$