A CHARACTERIZATION OF PARACOMPACTNESS IN TERMS OF FILTERBASES

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In elementary courses in general topology, students study filterbases and encounter characterizations of compact spaces, Lindelöf spaces, countably compact spaces, etc. in terms of filterbases. Later, when studying paracompact spaces, students ask for a characterization of such spaces in terms of filterbases. The purpose of this note is to make available such a characterization. We assume familiarity with filter subbases and filterbases from N. Bourbaki [1] and all spaces are topological spaces. Recall that a collection of subsets of a space is locally finite if each point in the space has a neighborhood which has nonempty intersection with at most finitely many members of the collection. A family of subsets Ω refines a family of subsets Γ if each $A \in \Omega$ satisfies $A \subset B$ for some $B \in \Gamma$. A space X is paracompact if for each covering Λ of X by open sets, there is a covering of X by open sets which is locally finite and which refines Λ [2].

Definition 1. In a space X a collection of subsets Ω of X is *locally ultimately dominating (l.u.d.)* if for each $x \in X$ there is an open set about x contained in all but finitely many elements of Ω.

Definition 2. A filterbase Ω on a space is of type $\mathcal{P}$ if each l.u.d. filter subbase coarser than Ω has nonempty adherence (we say that a filter subbase Ω is coarser than a filterbase Γ if each element of Ω contains an element of Γ).

We offer the following characterization of paracompact spaces. Although, customarily, the Hausdorff separation axiom is assumed in the definition of paracompact spaces, no separation axioms are assumed in this note. If Ω is a filter subbase on a space the adherence of Ω will be denoted by adh Ω.

Theorem 1. A space is paracompact if and only if every filterbase of type $\mathcal{P}$ on the space has nonempty adherence.

Before giving the proof of Theorem 1 we state Lemma 1 without proof.

Lemma 1. A filterbase is of type $\mathcal{P}$ if and only if every coarser l.u.d. closed filter subbase has nonempty adherence.

Proof of Theorem 1. For the necessity part of the proof, let the space X be paracompact and let Ω be a filterbase on X such that adh $\Omega = \emptyset$. Then $\{X - \overline{F} : F \in \Omega\}$ is an open cover of X which has a locally finite open refinement κ. Then