FUN WITH THE $\sigma(\mathbf{n})$ FUNCTION

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1. Notation (Standard and Otherwise). The function $\sigma(n)$ is one of the basic number-theoretic arithmetic functions. It is defined as:

$$\sigma(n) = \sum_{d|n} d.$$

Some values of $\sigma(n)$ for small n can be found in [2, sequence A000203]. (Note: It is known that $\sigma(n)$ is also multiplicative, i.e., if j and k have no factors in common other than 1, $\sigma(jk) = \sigma(j)\sigma(k)$.) The Dirichlet convolution of two arithmetic functions $f(n)$ and $g(n)$, itself a function of n, is defined as follows.

$$f * g = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

We will use $\perp$ to denote relative primality.

2. Sigma-Primes. A number n is called *sigma-prime* if and only if $n \perp \sigma(n)$. The sigma-prime numbers below 100 can be found in [2, sequence A014567]. Two rather straightforward theorems are the following.

Theorem 1. All powers of primes are sigma-prime.

Theorem 2. No perfect numbers are sigma-prime.

To build this theory, we shall, in the time-honored tradition of mathematics, start with the simple examples and move up. If a number n is the product of two primes, say p and q, then $\sigma(n) = 1 + p + q + pq = \sigma(p)\sigma(q)$. Now, the only divisors of pq are p and q. Clearly, $p \perp \sigma(p)$. Thus, $p \perp \sigma(pq)$ if and only if $p \perp \sigma(q)$. Similarly, $q \perp \sigma(pq)$ if and only if $q \perp \sigma(p)$. Assuming $p < q$, we can see that (unless $p = 2$ and $q = 3$), $p + 1 < q$ and from that $p + 1 \perp q$. Note also that in the case of the above exception, $q + 1 \not\perp p$. Thus, we can generalize and say that $n = pq$ is sigma-prime if and only if $q + 1 \perp p$. This easily extends to the following theorem.

Theorem 3. If $n = p_1 p_2 \cdots p_k$, where $p_1 < p_2 < \cdots < p_k$, and each of $p_1, \ldots,$ p_k is a prime, then n is sigma-prime if and only if $p_i \perp 1 + p_j$ whenever $i < j$.