UNIQUENESS OF ROW ECHELON FORM

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The following definition is familiar to every student having had any exposure to linear algebra or matrix theory:

Definition. Let $A = (a_{ij})$ be an $n \times m$ matrix over a field k. We say that A is in *row echelon form* if:

(a) The first non-zero entry of each row is 1.
(b) Rows $1, 2, \ldots, r$ are the non-zero rows and rows $r+1, r+2, \ldots n$ are zero rows.
(c) If $a_{1,i_1}, a_{2,i_2}, \ldots, a_{r,i_r}$ are the first non-zero entries in rows $1, 2, \ldots r$, respectively, then $i_1 < i_2 < \cdots < i_r$.
(d) For all $j = 1, 2, \ldots, r$ and for all $k < j$, we have $a_{k,i_j} = 0$.

Students learn that by using *Gauss-Jordan elimination,* any matrix is row-equivalent to a matrix in row echelon form. However, while linear algebra or matrix theory textbooks often assert the uniqueness of row echelon form of a matrix, relatively few actually provide a proof. The most commonly found proof [2,3,4] shows first that the columns $i_1, i_2, \ldots, i_r$ carrying the initial 1s of (a) above are uniquely determined, and then goes on to show that the remaining entries all coincide. A second proof, found in [1], interprets the matrix as the matrix of a linear transformation $T: U \to V$, where row equivalence is manifested through changes of ordered bases in V. Furthermore, if the matrix is in row echelon form, then the representing ordered basis in V is necessarily uniquely determined by T and a fixed ordered basis of U.

While neither proof above is difficult, both are "microscopic," involving a close scrutiny of matrix entries. The present approach, on the other hand, is based on more holistic properties of matrix products.

The key ingredient of the present approach is to use the notion of *Hermite normal form,* defined below.

Definition. Let $A = (a_{ij})$ be an $n \times n$ matrix, i.e., a *square matrix* and assume that

1. The first non-zero entry of each row is 1.
2. The first non-zero entry of each row is on the diagonal.
3. If $a_{jj} \neq 0$, then for all $k < j$ we have $a_{kj} = 0$.

Then we say that the matrix A is in *Hermite normal form.* Note that unlike a matrix in row echelon form, a matrix in Hermite normal form may have zero rows interspersed among the non-zero rows. However, note that by a permutation of