A PROOF OF JENSEN'S INEQUALITY

Joseph Bennish

Jensen's inequality is one of the most important inequalities in all of mathematics. The version of this inequality for sums states that for any convex function $\phi(x)$ on an interval (a, b), the inequality

$$\phi(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \leq p_1\phi(x_1) + p_2\phi(x_2) + \cdots + p_n\phi(x_n), \qquad (1)$$

holds for any positive numbers $p_1, \ldots, p_n$ satisfying $p_1 + \cdots + p_n = 1$, and any numbers $x_1, \ldots, x_n$ in (a, b). Setting $n = 2$ in the above inequality gives the analytic definition of a convex function. Geometrically, convexity asserts that the curve lies on or below the whole chord. The purpose of this note is to present a short and simple proof of Jensen's inequality, both for sums and integrals, which could not be located in the literature by the author.

Jensen's inequality for sums will be proved assuming differentiability of ϕ. Afterwards, Jensen's inequality for integrals will be proved in complete generality. The proof of (1) uses the well-known property of differentiable convex functions that its derivative is an increasing function. Put

$$S = \sum_{i=1}^{n} p_i\phi(x_i) - \phi(A), \quad \text{where } A = \sum_{i=1}^{n} p_ix_i.$$

Jensen's inequality becomes $S \geq 0$. We have

$$S = \sum_{i=1}^{n} p_i\{\phi(x_i) - \phi(A)\} = \sum_{i=1}^{n} p_i \int_A^{x_i} \phi'(x)dx.$$

If $x_i \geq A$, the above integral is bounded below by $\phi'(A)(x_i - A)$. It is easily checked that this bound also holds in the case $x_i < A$. Thus,

$$S \geq \sum_{i=1}^{n} p_i\phi'(A)(x_i - A) = \phi'(A)\left(\sum_{i=1}^{n} p_ix_i - A\right) = \phi'(A)(A - A) = 0.$$