SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

21. [1990, 80; 1991, 95] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find distinct positive integers, a, b, c, d such that

$$a+b+c+d+abcd = ab+bc+ca+ad+bd+cd+abc+abd+acd+bcd.$$

Solution by Les Reid, Southwest Missouri State University, Springfield, Missouri.

More generally, we will find all positive integer solutions to

$$a+b+c+d+abcd = abc+abd+acd+bcd+ab+ac+ad+bc+bd+cd.$$

Rewriting as

$$abcd-abc-abd-acd-bcd-ab-ac-ad-bc-cd+a+b+c+d=0$$

and noting that the left-hand side has the opposite parity of $(a-1)(b-1)(c-1)(d-1)$ [since their difference is $2(ab+ac+ad+bc+bd+cd-a-b-c-d)+1$], we see that $(a-1)(b-1)(c-1)(d-1)$ must be odd, hence a, b, c, and d must all be even.

Since our original equation is symmetric in the variables, we may assume without loss of generality that $0 < d \leq c \leq b \leq a$.

If $d=2$ and $c=2$, our equation is equivalent to $ab+7a+7b=0$ which has no positive solutions.

If $d=2$ and $c=4$, our equation may be rewritten as $(a-13)(b-13)=171$. Since 171 can be factored as $171\cdot 1$, $57\cdot 3$, $19\cdot 9$, we obtain the solutions $a=184$, $b=14$; $a=70$, $b=16$; $a=32$, $b=22$.

If $d=2$ and $c=6$, our equation may be rewritten as $(3a-19)(3b-19)=373$. Since 373 is prime, the only possible factorization is as $373\cdot 1$, but this yields non-integer values for a and b.

If $d=2$ and $c=8$, our equation becomes $5ab-25a-25b=6$, which is clearly impossible since the left-hand side is not divisible by 5.