A NONHOMOLOGICAL PROOF OF SEMIPERFECTNESS IN MATRIX RINGS

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Introduction. Let R be an associative ring with unit. An element e of R is said to be an *idempotent* if $e^2 = e$. Two idempotents e, f of R are said to be *orthogonal* if $ef = fe = 0$. A nonzero idempotent e of R is said to be *primitive* if it cannot be written as the sum of two nonzero orthogonal idempotents. If e is an idempotent of R such that eRe is a local ring, that is eRe has exactly one maximal ideal, then e is said to be *local*. It is known (see [2] for example) that every local idempotent is primitive. However, the converse is not necessarily true. For example, 1 is a primitive but not local idempotent of $\mathbb{Z}$. For an ideal I of R, we say that idempotents of R/I can be *lifted* to R if for every idempotent $u + I \in R/I$, there exists an idempotent $e^2 = e \in R$ such that $e - u \in I$.

Denote the Jacobson radical of R by $J(R)$ and the ring of $n \times n$ matrices over R by $M_n(R)$. R is said to be *semiperfect* if $R/J(R)$ is Artinian and idempotents of $R/J(R)$ can be lifted to R. It has been shown by Kaye [1] via the Morita Duality Theorem that R is semiperfect if and only if $M_n(R)$ is semiperfect. The purpose of this paper is to give a nonhomological proof of this result.

All rings considered in this paper are assumed to be associative with unit.

1. Some Preliminaries. We first state the following result by B. J. Mueller [3] on conditions on a ring which are equivalent to being semiperfect.

Theorem 1.1. Let R be a ring. The following conditions are equivalent:

(i) R is semiperfect;

(ii) Every primitive idempotent of R is local and there is no infinite set of orthogonal idempotents in R;

(iii) The unit $1 \in R$ is the finite sum of some orthogonal local idempotents.

In what follows, for i, $j = 1, \ldots, n$, we let $E_{ij} = (e_{rs})$ denote the $n \times n$ matrix over R such that

$$e_{rs} = \begin{cases} 1 & \text{if } (r, s) = (i, j) \\ 0 & \text{if } (r, s) \neq (i, j) \end{cases}, \quad r, s = 1, \ldots, n.$$

Proposition 1.2. Let R be a ring. If e is a primitive idempotent of R, then eE_{tt} is a primitive idempotent of $M_n(R)$ for $t = 1, \ldots, n$.