INDEX OF A SUBGROUP OF AN ABELIAN GROUP

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Introduction. Lemma 1 of [2] states that if an abelian group G is a union of cosets $H_1 + a_1$, $H_2 + a_2$, $\ldots$, $H_n + a_n$, where the H_i's are subgroups of G and the a_i's are elements of G, then the index of one of these subgroups is finite. In that paper nothing was said about an upper bound for the index of that subgroup. In this paper, under the hypothesis of Lemma 1 of [2], we show that, for $n \geq 3$, the index of one of those subgroups is at most $2^{2(n-2)} - 2^{n-2} + 1$. The above conclusion is not true for nonabelian groups. It is easy to see that if

$$G = \bigcup_{i=1}^{2} (H_i + a_i)$$

and the H_i's are proper subgroups of G, then $H_1 = H_2$. Further, we show that if

$$G = \bigcup_{i=1}^{3} (H_i + a_i)$$

and the H_i's are proper subgroups of G, then either the index of one of these subgroups is two or all these subgroups are identical, and hence the index of one of these subgroups is at most three.

Notation. If A and B are subsets of an abelian group $(G, +)$, then $A + B$ and $A - B$ stand for $\{x + y : x \in A \text{ and } y \in B\}$ and $\{x - y : x \in A \text{ and } y \in B\}$, respectively. For $g \in G$, $A + \{g\}$ is denoted by $A + g$. The set $A \setminus B$ denotes $\{x \in A : x \notin B\}$.

Lemma 1. Let a and b be elements of an abelian group $(G, +)$. Suppose that H and K are proper subgroups of G such that H is not contained in K, and K is not contained in H. Then there exists a subset M of G of cardinality four such that, for every $g \in G$, the intersection of the set $M + g$ and the set $G \setminus ((H + a) \cup (K + b))$ is nonempty.

Proof. Let $h \in H \setminus K$, $k \in K \setminus H$, and $M = \{0,\ h,\ k,\ h + k\}$. Suppose, to the contrary, that $M + g \subseteq (H + a) \cup (K + b)$ for some $g \in G$. Without loss of generality, we may assume that $g \in H + a$. If $k + g \in H + a$, then $k = (k + g) - g \in$