DEMOIVRE'S FORMULA TO THE RESCUE

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1. Introduction. Euler's formulas

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{1}$$

and DeMoivre's theorem

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx \tag{2}$$

make an inseparable and elegant team. A few months ago, we spoke on this theme in San Antonio [1] and it was a real treat to read papers [2] and [3], indeed. Article [3] uses Euler's formulas to express the integer powers of sine and cosine as trigonometric polynomials (finite linear combinations of sines and cosines of multiple angles) and thus, to calculate important trigonometric integrals. In fact, a lot of earlier works, including [4] and [5], accomplished the same act and, we trust, the method has fascinated us since the times of Euler and Laplace. Then, why don't our calculus texts accept this technique of renowned experts? The reason, in our opinion, is quite clear: Euler's formulas look very nice and sleek, but their theory is both complex and deep.

In this connection, let us quote what the *Monthly* editorial [6] once wrote: "Judging from the volume of mail addressed to *Classroom Notes* it appears that one of the greatest mysteries of undergraduate mathematics is the equation of Euler $e^{ix} = \cos x + i \sin x$. The objective of these correspondents is to develop this formula without the use of infinite series; and judging from the desperate devices employed by these writers in seeking to attain this end, it is highly desirable that a rigorous simple proof of this formula be available .... The first point to be emphasized is that the expression e^{ix} has to be *defined*, and that certain properties must be ascribed to it. Otherwise any proof falls to the ground. Rigorous treatments of this appear in the classical literature; for example, see G. H. Hardy, *Pure Mathematics*, p. 409 (fifth edition), or E. T. Whittaker and G. N. Watson, *Modern Analysis*, p. 581 (fourth edition). Since these have more general objectives in view it may be complained that they are too complicated for the purpose of defining the relatively simple expression e^{ix}. If, on the other hand, one wishes simplicity, he may directly