A NOTE ON THE ISOPERIMETRIC INEQUALITY

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In most calculus books students are introduced to an isoperimetric theorem in the following form. Among all rectangles with a given perimeter the square has the largest area [5]. However, this isoperimetric theorem for rectangles can be proved easily in an algebraic class using quadratic functions. The following theorem might be more appropriate for a calculus class. Among all quadrilaterals with a given perimeter and a given side, the trapezoid with the other sides of equal length, and of equal angles between them has the largest area. This isoperimetric theorem for quadrilaterals has a nice application, an isoperimetric theorem for n-polygons, i.e. polygons with n vertices. Among all n-polygons with a given side and a given perimeter, the n-polygon with the maximum area is inscribed in a circle and has all other sides of equal length and of equal angles between them where the existence of such an n-polygon is obtained from the general result that a continuous function on a bounded and closed subset of an Euclidean plane, $\mathbb{R}^N$, attains a maximum value.

In this note proofs of the above theorems are presented and the theorems are then utilized along with inscribed polygons to obtain a proof for the following isoperimetric theorem for simple closed curves. Let S be a closed curve formed by a circular arc of length s together with its chord of length ℓ. Then any simple closed curve Σ formed by a curve of length s together with a line segment of length ℓ satisfies the inequality $A(\Sigma) \leq A(S)$ where $A(\sigma)$ denotes the area enclosed by the simple closed curve σ and the equality holds if and only if Σ coincides with S. As a corollary we obtain the isoperimetric theorem for simple closed curves [2]. Any simple closed curve Σ with length s satisfies the inequality $4\pi A(\Sigma) \leq s^2$, with equality if and only if Σ is a circle.

In his paper [4] "The Isoperimetric Inequality," Professor Osserman obtains the following inequality for any n-polygon Σ with perimeter s.

$$\frac{s^2}{A(\Sigma)} \geq \frac{4}{n} \tan \frac{\pi}{n} > 4\pi.$$