GRAPHIC REPRESENTATIONS FOR ASSOCIATIVE ALGEBRAS

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1. Introduction. In mathematics, many problems become easier to solve when it is possible to have some kind of graphic representation either of the problem itself, or of the objects which are involved in it. By that way, the searcher (or the student) can get a geometric intuition of what he/she is studying. Associative algebras arise in many fields of mathematics and their study is a very live field of research. Therefore we present here a construction of some graphs associated to an associative algebra with unit which provide this intuition.

2. Peirce Decomposition of an Algebra. Let A be an associative algebra with unit 1 over a sufficiently large algebraically closed field K. In this note, we will call A an algebra. In fact, A is a vector space over K. Since we have taken K sufficiently large, a theorem of Wedderburn implies that, as a vector space, A is the direct sum of a (non-unique) subalgebra S and an ideal J, i.e. $A = S \oplus J$. The subalgebra S is a direct sum of matrix blocks $S \oplus_i M_{n_i}(K)$, and J is an ideal, i.e. $aJ \subseteq J$ and $Ja \subseteq J$ for any $a \in A$. The subalgebra S is called a separable subalgebra of A and J is called the radical of A.

If e is an idempotent element in A, i.e. $e^2 = e$, then every element in A can be written in the form $a = eae + ea(1-e) + (1-e)ae + (1-e)a(1-e)$. Moreover, if we write $xAy = \{xay\ ;\ a \in A\}$, it is clear that we have the following decomposition of the algebra A:

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e).$$

This decomposition is called the two-sided Peirce decomposition of A with respect to e. The subspaces eAe, $eA(1-e)$, $(1-e)Ae$ and $(1-e)A(1-e)$ are called the Peirce components of A with respect to e.

Proposition 2.1. Let A be an arbitrary algebra. If J is the radical of A, then $eJe = eAe \cap J$, $eJ(1-e) = eA(1-e) \cap J$, $(1-e)Je = (1-e)Ae \cap J$, $(1-e)J(1-e) = (1-e)A(1-e) \cap J$.

For a proof. see [5].

Now let A be an algebra. Two idempotents e and f are orthogonal if $ef = fe = 0$; an idempotent e is primitive if it cannot be written as the sum of two non-zero idempotents. A family $e_1, e_2, \ldots, e_r$ of idempotents is a family of orthogonal