DIRECTED GRAPHS, MAGIC SQUARES, AND GROTHENDIECK TOPOLOGIES

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1. Introduction. It is well-known that a graph can be represented by a square matrix by considering its adjacency matrix. One of the goals of this paper is to give an algebraic description of such a correspondence for directed graphs.

A directed graph can be viewed as an order pair (α, β) of mappings from the set of directed edges to the set of vertices in such a way that a directed edge e is the one with the initial vertex $\alpha(e)$ and the terminal vertex $\beta(e)$ [4]. Thus, with an appropriate morphism, we can consider the category of directed graphs whose objects are ordered pairs of mappings of finite sets. More precisely, we consider the category of directed graphs with a fixed set X of vertices whose objects are viewed as the set of ordered pairs (α, β) of mappings $\alpha, \beta: Y \to X$ from various finite sets Y to the given set X. We extend the set of isomorphism classes of the objects in this category to a set which has a ring structure and prove that the resulting ring is isomorphic to the ring of $m \times m$ integral matrices, where m is the number of elements in X (Theorem 1). We also consider a subring of this ring corresponding to regular digraphs and show that it is isomorphic to the ring of generalized magic squares (Theorem 2).

Given a finite set X, the category of single mappings $\phi: Z \to X$ can be regarded as a Grothendieck topology on the category of finite sets [2, 3]. In Section 7, we discuss the action of the ring associated to directed graphs above on the objects of this category.

2. Directed Graphs. A directed graph $G = (V, E)$ consists of a finite set V of vertices and a finite set E of edges, where each edge is an ordered pair of vertices. Let $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_n\}$. If $e_l = (v_i, v_j)$, then v_i is called the initial vertex and v_j is called the terminal vertex of e_l.

To each directed graph $G = (V, E)$, we can associate an ordered pair (α, β) of mappings $\alpha, \beta: E \to V$ such that, for each edge $e \in E$, $\alpha(e)$ is the initial vertex and $\beta(e)$ is the terminal vertex of e [4]. Conversely, to each ordered pair (f, g) of mappings $f, g: Y \to X$ of finite sets, we can associate a directed graph such that X is the set of vertices, Y is the set of edges, and each $y \in Y$ has the initial vertex