AN EXTENSION OF DINI'S LEMMA TO NETS OF FUNCTIONS INTO UNIFORM SPACES

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A classical and important lemma of Dini in analysis states:

Theorem D. If $\{f_n\}$ is a sequence of real-valued continuous functions on $[0,1]$ such that $\{f_n(x)\}$ is a non-increasing sequence converging to 0 for each $x \in [0,1]$ then $f_n \to 0$ uniformly.

In this note we extend this lemma to nets of functions mapping into uniform spaces. The following readily established results are used as patterns to formulate and prove the extension.

(1°) A non-increasing sequence $\{x_n\}$ of nonnegative reals converges to 0 if and only if for each $\epsilon > 0$, $\emptyset \neq \{n : x_n - 0 < \epsilon\} \subset \{n : x_k - 0 < \epsilon \text{ for every } k \geq n\}$.

An equivalent form of Theorem D is

(2°) If $\{f_n\}$ is a sequence of real-valued continuous functions on $[0,1]$ such that $\{|f_n(x)|\}$ is a non-increasing sequence converging to 0 for each $x \in [0,1]$ then $f_n \to 0$ uniformly.

These observations motivate the following definitions.

Definition 1. If X is a uniform space with uniformity $\mathcal{U}$ we say a net $\{x_n\}_\Lambda$ converges to $x \in X$ in a monotone fashion if for each $U \in \mathcal{U}$, $\emptyset \neq \{n : (x_n, x) \in U\} \subset \{n : (x_k, x) \in U \text{ for every } k \in \Lambda \text{ with } k \geq n\}$.

Definition 2. Let Y be a uniform space with uniformity $\mathcal{U}$, X be a nonempty set, and $f\colon X \to Y$ a function. We say that a net $\{f_n\}_\Lambda$ of functions from X to Y is pointwise monotonically convergent to f if $\{f_n(x)\}_\Lambda$ converges to $f(x)$ in a monotone fashion for each $x \in X$.

A net $\{f_n(x)\}_\Lambda$ from a topological space X to a uniform space Y with uniformity $\mathcal{U}$ converges to $f\colon X \to Y$ uniformly if for each $U \in \mathcal{U}$, $(f_n(x), f(x)) \in U$ ultimately. For facts on topological and uniform spaces used here without definition see [1].

We are now prepared to present our extension of Dini's lemma.

Theorem. Let X be a compact space and let Y be a uniform space with uniformity $\mathcal{U}$. Let $\{f_n\}_\Lambda$ be a net of continuous functions from X to Y. If $\{f_n\}_\Lambda$ is pointwise monotonically convergent to the continuous function $f\colon X \to Y$ then $\{f_n\}_\Lambda$ converges to f uniformly.