NONEMPTY INTERSECTIONS OF MIDDLE α CANTOR SETS

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Periodically, one is lucky enough to be able to find an interesting result from current mathematical research that is accessible to undergraduate students. This paper describes such a result from the field of Dynamical Systems. Here we will be exploring the nonempty intersections of two middle α Cantor sets as they are translated across one another. We will present criteria for which the intersection between two such Cantor sets is always nonempty as they are translated across one another. This fact has generated much interest and discussion in some of my upper division classes for mathematics majors and perhaps it can do the same for others.

In the past, most students of mathematics have been introduced to Cantor sets in an introductory course of real analysis. In introductory real analysis, the middle third Cantor set is explored as an example of an uncountable, closed set which contains no interior points or isolated points and has Lebesgue measure zero. More recently, with the popularization of fractals, the middle third Cantor set has become a standard example of a self-similar set. The middle third Cantor set is merely a specific example of a middle α Cantor set, where α has been set to $1/3$.

While the result that we will present here is interesting in its own right, it is also interesting to know some of the motivation for studying intersecting Cantor sets. Earlier we noted that this problem is connected to the discipline of Dynamical Systems. A brief history of why intersections of Cantor sets are important in Dynamical Systems starts in the late 1800's with Poincaré. Poincaré had identified a problem common to understanding many nonlinear dynamical systems, i.e, how to describe changes in the system as a homoclinic bifurcation takes place. It is known that as such a bifurcation takes place, the behavior of a deterministic dynamical system can change dramatically from an easy to understand stable system to a completely chaotic system.

Over the past 20 years, there has been much work done in understanding homoclinic bifurcations (see [9] for a recent overview of the subject). Several major theories have been built around homoclinic bifurcations: omega explosions [8], infinitely many co-existing sinks [1, 7, 10], and antimonotonicity [4]. Each of these theories rely on knowledge of how certain stable and unstable manifolds intersect