SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

117. [1998, 117] *Proposed by Kenneth B. Davenport, 301 Morea Road, Box 491, Frackville, Pennsylvania.*

(a) Prove that

$$\int_0^\infty \frac{1}{1+x^3}\cdot\frac{1}{1+r^3x^3}\cdots\frac{1}{1+(r^nx)^3}dx$$

$$= \frac{2\pi}{3\sqrt{3}}\cdot\frac{(1-r^2)(1-r^5)\cdots(1-r^{3n-1})}{(1-r^3)(1-r^6)\cdots(1-r^{3n})}.$$

(b) Prove that

$$\int_0^\infty \frac{1}{1+x^4}\cdot\frac{1}{1+r^4x^4}\cdots\frac{1}{1+(r^nx)^4}dx$$

$$= \frac{\pi}{2\sqrt{2}}\cdot\frac{(1-r^3)(1-r^7)\cdots(1-r^{4n-1})}{(1-r^4)(1-r^8)\cdots(1-r^{4n})}.$$

Solution by Larry Eifler and Jerry Masuda, University of Missouri-Kansas City, Missouri.

Fix $m > 1$ and let $r > 0$. For $n = 0, 1, 2, \ldots$ and for $x \geq 0$, define

$$\phi_n(r,x) = \prod_{k=0}^{n} \frac{1}{1+(r^kx)^m}$$

$$= \frac{1}{1+x^m}\cdot\frac{1}{1+(rx)^m}\cdots\frac{1}{(r^nx)^m}.$$

Set

$$I_n(r) = \int_0^\infty \phi_n(r,x)dx \qquad \text{for } n = 0, 1, 2, \ldots.$$