SOME REPRESENTATIONS OF $\zeta(3)$

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1. Introduction. The Riemann zeta function ζ is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

for each complex number z with real part Re $z > 1$. In this paper we only concentrate on $\zeta(3)$. R. Apéry [1] proved that $\zeta(3)$ is an irrational number using the formula

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}. \tag{1}$$

Motivated by Apéry's proof, F. Beukers [2] later gave a shorter proof of the irrationality of $\zeta(3)$ by means of double and triple integrals. Beukers' proof hinged on his formula

$$\zeta(3) = \int_0^1 \int_0^1 \frac{-\log xy}{1-xy}\, dx\, dy \tag{2}$$

where the integrals can be justified by replacing $\int_0^1$ with $\lim\limits_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon}$.

The value of $\zeta(3)$, however, remains unknown, let alone the values of ζ at other larger odd integers.

The formulas

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{2\pi\sqrt{3}+9}{27},$$

$$\sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \frac{\pi\sqrt{3}}{9},$$