A NOTE ON THE SUM $\sum 1/w_{k2^n}$

Stanley Rabinowitz

1. Historical Results. In 1974, Millin [13] published a problem stating that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}. \tag{1}$$

This spurred a flurry of activity: [1, 3, 4, 5, 6, 7, 8, 17]. Most investigators, however, overlooked the fact that Lucas studied such sums back in 1878. He showed in [11], equation (125), that if $k \neq 0$, then

$$\sum_{n=1}^{N} \frac{Q^{k2^{n-1}}}{u_{k2^n}} = \frac{Q^k u_{k(2^N-1)}}{u_k u_{k2^N}} \tag{2}$$

where u_n is a second order linear recurrence defined by

$$u_{n+2} = Pu_{n+1} - Qu_n, \quad u_0 = 0, \quad u_1 = 1.$$

If we use the identity $Q^{n-1}u_{m-n} = u_n u_{m-1} - u_m u_{n-1}$, we can express formula (2) in the form

$$\sum_{n=1}^{N} \frac{Q^{k2^{n-1}}}{u_{k2^n}} = Q\left[\frac{u_{k2^N-1}}{u_{k2^N}} - \frac{u_{k-1}}{u_k}\right]. \tag{3}$$

If $Q = -1$, as is the case for Fibonacci, Lucas, and Pell numbers, then equation (3) becomes

$$\sum_{n=0}^{N} \frac{1}{u_{k2^n}} = \frac{1+u_{k-1}}{u_k} + \frac{1-(-1)^k}{u_{2k}} - \frac{u_{k2^N-1}}{u_{k2^N}} \tag{4}$$