COMPOSITE RULES FOR IMPROPER INTEGRALS

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1. Introduction. In this note we consider how composite rules for approximate integration may be applied to certain types of improper integrals. Recall that composite rules are based on piecewise polynomials, e.g. the composite trapezoid rule is based on piecewise linear interpolation.

From approximation theory we know that if p_n is the piecewise linear function which interpolates a function $f \in C^2[0,1]$ at the points $\{i/n \;:\; 0 \le i \le n\}$ then

$$\|p_n - f\|_\infty = O(n^{-2}).$$

In [1] de Boor considers what happens when f is taken to be the square root function. The rate of convergence deteriorates:

$$\|p_n - f\|_\infty = O(n^{-1/2}).$$

However, if, rather than using a uniform mesh, we concentrate more mesh points toward the origin, then the optimal rate of convergence $O(n^{-2})$ can be recovered. In particular, if $p_{q,n}$ is the piecewise linear polynomial interpolating $f(x) = x^{1/2}$ at the points

$$x_i := \left(\frac{i}{n}\right)^q, \; 0 \le i \le n, \tag{1}$$

then

$$\|p_{q,n} - f\|_\infty = O(n^{-2}),$$

provided that $q \ge 4$. It follows that if $q \ge 4$ then the optimal order of convergence holds for the corresponding composite trapezoid rule, i.e.

$$\left| \int_0^1 p_{q,n}(x) - f(x)dx \right| = O(n^{-2}). \tag{2}$$

From numerical experiments however, it appears that the same rate of convergence holds for q as small as $4/3$. In the following figure

$$t = \log_2\left(\left| \int_0^1 p_{q,n}(x) - f(x)dx \right| / \left| \int_0^1 p_{q,2n}(x) - f(x)dx \right| \right),$$