SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

81. [1995, 87] *Proposed by J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin.*

Let ABC be a triangle with sides a, b, and c. Let K be the area of triangle ABC and s be the semi-perimeter of ABC.

(a) Prove that

$$\frac{K}{\tan\frac{A}{2}} + K\tan\frac{A}{2} = bc.$$

(b) Prove that

$$\frac{K}{s\tan\frac{A}{2}} + s = b + c.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

We begin by noting that

$$\tan\frac{A}{2} = \frac{r}{s-a} = \frac{rs}{s(s-a)} = \frac{K}{s(s-a)}$$

where r is the inradius of triangle ABC.

(a) Thus, by Heron's Formula and some algebra,

$$\begin{aligned}\frac{K}{\tan\frac{A}{2}} + K\tan\frac{A}{2} &= s(s-a) + \frac{K^2}{s(s-a)}\\ &= s(s-a) + \frac{s(s-a)(s-b)(s-c)}{s(s-a)}\\ &= s(s-a) + (s-b)(s-c) = 2s^2 - s(a+b+c) + bc\\ &= 2s^2 - s(2s) + bc = bc.\end{aligned}$$