SOME CHARACTERIZATIONS OF PERFECT NUMBERS

Joe Flowers

Northeast Missouri State University

In this paper, several (perfectly many) characterizations of perfect numbers are presented. A positive integer n is perfect (by definition) if and only if $\sigma(n) = 2n$ where $\sigma(n) = \sum_{d|n} d$ is the sum of all (positive) divisors of n. Replacing d with n/d in the equation $\sum_{d|n} d = 2n$ yields the alternative definition: n is perfect if and only if $\sum_{d|n} 1/d = 2$.

Two very old unsolved problems concerning perfect numbers are:

(i) Do infinitely many perfect numbers exist?

(ii) Do *any* odd perfect numbers exist?

The function σ is an example of a multiplicative function since it has the property $\sigma(mn) = \sigma(m)\sigma(n)$ whenever $\gcd(m, n) = 1$. Other multiplicative functions which appear in the paper are: τ where $\tau(n) = \sum_{d|n} 1$ is the number of divisors of n; Euler's function ϕ where $\phi(n)$ is the number of integers x such that $1 \leq x \leq n$ and $\gcd(n, x) = 1$; E where $E(n) = n$; U where $U(n) = 1$ for all n; the Moebiüs function μ where $\mu(1) = 1$, $\mu(n) = 0$ if $p^2|n$ for some prime p, $\mu(n) = (-1)^\alpha$ if n is the product of α distinct primes; i where $i(1) = 1$, $i(n) = 0$ for $n > 1$; r where $r(n) = 1/n$; h where $h(n) = n^2$; f where $f(n) = \sigma(n^2)$.

Each of the following 28 formulas is a necessary and sufficient condition for n to be perfect. For the most part, each formula is obtained by substituting $2n$ for $\sigma(n)$ in a more general formula involving σ.

The first three formulas all follow from LaGrange's identity,

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 = \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2$$

(see [1], page 27). The notation $\sum_{e<d}$ denotes the sum over pairs e and d of divisors of n:

(1) $$\tau(n) \sum_{d|n} d^2 = 4n^2 + \sum_{e<d} (d - e)^2;$$