SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

69. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let G be a group such that whenever $g_1, g_2, g_3 \in G$ and $g_1g_2 = g_3g_1$, then $g_2 = g_3$. Show that:

(a) If G has two elements of order 2, then G must contain the Klein four group.

(b) The set $H = \{g \mid g^k = 1, \text{where } k \text{ is some integer}\}$ is a subgroup of G.

Composite solution by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; and the proposer.

First we shall prove that G is abelian. Let a and b be elements of G. Since $b(ab) = (ba)b$, it follows from the given condition that $ab = ba$ for all $a, b \in G$. Thus, G is abelian.

(a) Suppose a and b are distinct elements of G of order 2. If $ab = 1$, then $a = a \cdot 1 = aab = 1 \cdot b = b$. Hence, $ab \neq 1$ and since

$$(ab)^2 = a^2b^2 = 1,$$

the order of ab is 2. Finally, ab is distinct from a and b because if $ab = a$ or $ab = b$, then $b = 1$ or $a = 1$. Now it can be easily seen that the set consisting of the elements $1, a, b$, and ab is isomorphic to the Klein four group.

Comment by the editor. The editor is responsible for some poor wording in part (b) of this problem. As a result, two different solutions to the (b) part of this problem were given. The editor should have worded part (b) in one of the following ways.

(b1) Let k be a fixed integer. The set

$$H = \{g \mid g^k = 1\}$$

is a subgroup of G.

Solution by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and the proposer.

Clearly the identity element is in H. Now, suppose x and y are in H. It is enough to show that

$$(xy^{-1})^k = 1.$$