INDEPENDENT RANDOM VARIABLES ON THE UNIT INTERVAL

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Let the probability of a subset of $[0,1]$ be given by its Lebesgue measure, i.e., the uniform distribution. In this paper we relate independent random variables, which are continuous functions, to space filling curves. In modern terminology, a random variable is a real valued measurable function defined on a probability space. A collection V of random variables is said to be an independent collection if, for any natural number n and for any collection of functions $\{f_1, f_2, \ldots, f_n\} \subset V$ and Borel subsets (equivalently, open intervals) $A_1, A_2, \ldots, A_n$ of $\mathbb{R}$, we have

$$(1) \qquad \Pr\left(\bigcap_{i=1}^{n}\{x \mid f_i(x) \in A_i\}\right) = \prod_{i=1}^{n} \Pr\left(\{x \mid f_i(x) \in A_i\}\right).$$

We are interested in considering random variables defined on the probability space consisting of the unit interval with the probability of a set given by its Lebesgue measure. A classical example of a collection of independent random variables defined on $[0,1]$ is that of the Rademacher functions $\{f_n(x)\}_{n=1}^{\infty}$ where

$$f_n(x) = 1, \text{ if } x \in [m/2^n, (m+1)/2^n)$$

with m even and

$$f_n(x) = -1, \text{ if } x \in [m/2^n, (m+1)/2^n)$$

with m odd.

Considerable work has been done studying general measurable functions which are independent on $[0,1]$ and on the intervals $[0,\infty)$ and $\mathbb{R} = (-\infty, \infty)$.

Many references to early work on this subject as well as the papers themselves, may be found in [4]. The possibility that such functions be continuous began with the observation that the coordinate functions of the Peano curve, which maps $[0,1]$ continuously into the unit square, are independent. (See [1] or [2] for recently discovered properties and a lucid description of the Peano curve.) Sierpinski [3] showed that if $x(t)$ and $y(t)$ are the coordinate functions of the Peano curve, then $f_n(t) = x(y^n(t))$, $n = 0, 1, 2, \ldots$ where $y^0(t) = t$, $y^n(t) = y(y^{n-1}(t))$ are independent and map $[0,1]$ onto $[0,1]^\omega$.