SET VALUED MAPPINGS ON THE REALS

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1. Introduction. Let c be a fixed positive real number and let k be a cardinal smaller than $2^{\aleph_0}$. Let f, g, and h be functions from the set of all real numbers, $\mathbb{R}$, into the family of all subsets of $\mathbb{R}$, $P(\mathbb{R})$, such that, for every r in $\mathbb{R}$, $f(r)$ is a closed set of Lebesgue measure less than c, $g(r)$ is a nowhere dense subset of $\mathbb{R}$, and the cardinality of $h(r)$ is smaller than k. Two real numbers x and y are said to be free (independent) for f, if $x \notin f(y)$ and $y \notin f(x)$. A set of real numbers is said to be free for f, if every pair of real numbers in this set is free. It is known that each of these functions f, g, and h admits an infinite free set (see [5], [1], and [3]). The purpose of this paper is to investigate free sets (for mappings) when the above mentioned conditions are mixed. For example, using some set-theoretic axioms, we show that the function $W : \mathbb{R} \to P(\mathbb{R})$, defined by $W(r) = g(r) \cup h(r)$, admits an infinite free set. But the function $W : \mathbb{R} \to P(\mathbb{R})$, defined by $W(r) = f(r) \cup g(r)$, need not admit an infinite free set.

Throughout this paper, the set of all real numbers, the set of all positive integers, and the family of all subsets of the reals are denoted by $\mathbb{R}$, $\mathbb{N}$, and $P(\mathbb{R})$ respectively. The cardinality of any set A is denoted by $|A|$. k^+ is the cardinal successor of k.

Definition 1. Two real numbers x and y are said to be free for a function $f : \mathbb{R} \to P(\mathbb{R})$ if $x \notin f(y)$ and $y \notin f(x)$. A set of real numbers is said to be free for f if every pair of real numbers in this set is free.

Philosophical work concerning free sets can be found in [2].

Definition 2. Any countable union of nowhere dense subsets of $\mathbb{R}$ is called a set of first category (or meager). Any subset of $\mathbb{R}$ that is not of first category is called a set of second category (non meager). The complement of a first category subset of $\mathbb{R}$ is called a residual set. A set M is everywhere of second category if $M \cap I$ is of second category for every non-empty open interval I.

Definition 3. Martin's Axiom states that no compact Hausdorff space with the ccc (recall that a topological space has the ccc means that it has no uncountable collection of pairwise disjoint open sets) can be the union of less than $2^{\aleph_0}$ nowhere dense subsets.

The following example shows that if f and g are functions from $\mathbb{R}$ into $P(\mathbb{R})$ admitting infinite free sets, then the function W, defined by $W(r) = f(r) \cup g(r)$, need not admit an infinite free set.