THE IDEAL STRUCTURE OF $\mathbf{Z} * \mathbf{Z}$

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1. Introduction. Let $\mathbb{Z}$ be the set of integers with usual addition and multiplication. Then the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ can be naturally made into a ring via the two operations componentwise addition and multiplication. We will denote this ring by $\mathbb{Z} \times \mathbb{Z}$.

However, there are other operations on the underlying set $\mathbb{Z} \times \mathbb{Z}$ which would make it into a ring. For example, consider the two operations given by,

$$(x, y) + (a, b) = (x + a, y + b)$$
$$(x, y) \cdot (a, b) = (xa, xb + ya + yb)$$

where x, y, a and b are elements of $\mathbb{Z}$. Then it can be shown that the set $\mathbb{Z} \times \mathbb{Z}$ with these operations forms a commutative ring with identity element $(1, 0)$. In this paper, we will denote this new ring by $\mathbb{Z} * \mathbb{Z}$, just to distinguish it from the usual Cartesian product ring $\mathbb{Z} \times \mathbb{Z}$.

The multiplication operation in $\mathbb{Z} * \mathbb{Z}$ seems to be rather unnatural, but it is the same as the multiplication considered in the well known Dorroh Extension Theorem. According to this theorem, any ring R can be embedded in a ring S with identity. To construct S, one would consider the set $\mathbb{Z} \times R$ and define two operations as,

$$(z_1, r_1) + (z_2, r_2) = (z_1 + z_2, r_1 + r_2)$$
$$(z_1, r_1) \cdot (z_2, r_2) = (z_1 z_2, z_1 r_2 + z_2 r_1 + r_1 r_2).$$

It can be shown that the set $\mathbb{Z} \times R$ with the above operations forms a ring with identity element $(1, 0)$. Then denoting this ring by $S = \mathbb{Z} * R$, one can show that the map $f \colon R \to S$ given by $f(r) = (0, r)$ is a ring monomorphism. For more details on Dorroh Extension Theorem the reader can refer to [2] and [4].

In view of this, our ring $\mathbb{Z} * \mathbb{Z}$ can be called "the Dorroh $\mathbb{Z}$ ring". A good question to ask would be, "what is the ideal structure of $\mathbb{Z} * \mathbb{Z}$?" Also of interest is the comparison of the ideal structure of $\mathbb{Z} * \mathbb{Z}$ to that of $\mathbb{Z} \times \mathbb{Z}$. Therefore, it is appropriate to start with some remarks on the old ring $\mathbb{Z} \times \mathbb{Z}$.