A NOTE ON MODULE HOMOMORPHISMS

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During a recent course in ring theory, I asked a student of mine to find examples of a ring and modules to show that the two conditions for a function to be a module homomorphism are independent of each other. Let R be a (not necessarily commutative) ring and M and N be left R-modules. A function $f\colon M \to N$ is said to be an R-module homomorphism if
(1) $f(x+y) = f(x) + f(y)$ for all $x, y \in M$ and
(2) $f(rx) = rf(x)$ for all $x \in M$, $r \in R$.

Let R be a noncommutative ring and fix $r \in R$ with r not in the center of R. Then $f\colon R \to R$ defined by $f(s) = rs$ is a function from the left R-module R to itself which satisfies condition (1). However, due to the noncommutativity, there exists $s, t \in R$ such that $f(ts) = rts \neq trs = tf(s)$. Thus (2) fails.

Likewise, we can find an example to satisfy (2) but not (1). Let $R = \mathbb{Z}_2$ and let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Define $f\colon M \to M$ by $f(0,0) = (0,0)$ and $f(x,y) = (1,1)$ if $(x,y) \neq (0,0)$. It is straight forward to check that (2) is satisfied. But $f[(1,0) + (0,1)] = f(1,1) = (1,1) \neq f(1,0) + f(0,1)$. Hence, (1) fails.

One question this note addresses is when does (2) imply (1)? The converse is more interesting if (2) implies (1), what can be said about M? Recall that M is a cyclic left R-module if there exists $x \in M$ with $Rx = M$.

Lemma 1. Suppose that R is a ring and N is any left R-module. If M is a cyclic left R-module and $f\colon M \to N$ is a function which satisfies (2), then f satisfies (1).

Proof. Choose $x \in M$ with $M = Rx$. Let $x_1, x_2 \in M$. Then there exists $r_1, r_2 \in R$ with $x_1 = r_1 x$ and $x_2 = r_2 x$. Thus, $f(x_1 + x_2) = f(r_1 x + r_2 x) = f[(r_1 + r_2)x] = (r_1 + r_2)f(x) = r_1 f(x) + r_2 f(x) = f(r_1 x) + f(r_2 x) = f(x_1) + f(x_2)$.

Thus, M being cyclic is a sufficient condition for (2) to imply (1). We now exhibit necessary conditions on M. In other words, if $f\colon M \to N$ is a function in which (2) implies (1), what can be said about M?

Suppose that F is a field and V is a vector space over F. The definition of a linear transformation from V to V is the same as for module homomorphisms. Finding examples